ON THE THEORY OF DIFFRACTION OF PLANE ELASTIC AND ELECTROMAGNETIC WAVES

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Problems of diffraction of plane elastic and electromagnetic waves at contours or bodies of arbitrary shape are considered. These problems are solved by a method developed in [1]. Problems of this type have been considered, for example, in papers of Sobolev [2], Filippov [3], Korbanskii [4] and others. The diffraction of plane waves at straight slits or infinitely sharp edges has been treated in [2,3] by a method which differs from the one expounded below.

Solutions of problems of diffraction at a circular cylinder and a sphere are given as examples.

1. Formulation of the general problem of diffraction of elastic waves and the solution of two-dimensional problems.

In solving two- and three-dimensional problems of diffraction of plane elastic waves we shall assume that the equations of motion of the elastic medium which surrounds the body are, in general, nonlinear. The elastic medium will be taken as isotropic. Nonlinearity of the physical type will be considered; i.e. we shall consider that the components of the stress tensor of the elastic medium depend nonlinearly on the components of the strain tensor. In other words, while accepting the hypothesis of small strains, we shall consider that the medium does not obey Hooke's law and we shall solve problems taking account of small nonlinear terms in the equations of motion of the elastic medium.

Let us assume, for instance, that the relation between the components of the stress tensor and those of the strain tensor have the form

$$
\begin{aligned}
\sigma_{xx} &= \lambda A_0 + 2\mu\,(1+\beta_1 A_0)\,\varepsilon_{xx} + \beta_2 A_0^2 - \beta_1 A_1 + \beta_3(\varepsilon_{xx}^2 + {}^1/_4\,\varepsilon_{xy}^2 + {}^1/_4\,\varepsilon_{xz}^2)\\
\sigma_{yy} &= \lambda A_0 + 2\mu\,(1+\beta_1 A_0)\,\varepsilon_{yy} + \beta_2 A_0^2 - \beta_1 A_1 + \beta_3(\varepsilon_{yy}^2 + {}^1/_4\,\varepsilon_{yz}^2 + {}^1/_4\,\varepsilon_{xy}^2)\\
\sigma_{zz} &= \lambda A_0 + 2\mu\,(1+\beta_1 A_0)\,\varepsilon_{zz} + \beta_2 A_0^2 - \beta_1 A_1 + \beta_3(\varepsilon_{zz}^2 + {}^1/_4\,\varepsilon_{xz}^2 + {}^1/_4\,\varepsilon_{yz}^2)\\
\sigma_{yx} &= \mu\,(1+\beta_1 A_0)\,\varepsilon_{xy} + \beta_3\,[(\varepsilon_{xx}+\varepsilon_{yy})\,\varepsilon_{xy} + {}^1/_2\,\varepsilon_{xz}\varepsilon_{yz}] \qquad (1.1)\\
\sigma_{yz} &= \mu\,(1+\beta_1 A_0)\,\varepsilon_{yz} + \beta_3\,[(\varepsilon_{zz}+\varepsilon_{yy})\,\varepsilon_{yz} + {}^1/_2\,\varepsilon_{xy}\varepsilon_{xz}]\\
\sigma_{zx} &= \mu\,(1+\beta_1 A_0)\,\varepsilon_{xz} + \beta_3\,[(\varepsilon_{xx}+\varepsilon_{zz})\,\varepsilon_{xz} + {}^1/_2\,\varepsilon_{yz}\varepsilon_{xy}]
\end{aligned}
$$

(Murnaghan's law of elasticity [5]), where A_0, A_1 and A_2 are the invariants of the strain tensor; λ, μ, β_1, β_2 and β_3 are elastic constants. For $\beta_1 = \beta_2 = \beta_3 = 0$, we obtain Hooke's law. In relations (1.1) the nonlinear terms are even functions of the components of the strain tensor. To obtain an odd relation, the following may be considered:

$$
\begin{aligned}
\sigma_{xx} &= \lambda A_0 + 2\mu\,(1+\gamma_1 A_0^2 + \gamma_2 A_1)\,\varepsilon_{xx} + \gamma_3 A_0^3 - (2\gamma_1 + \gamma_2 + \gamma_4)\,A_0 A_1 +\\
&\quad + \gamma_4 A_2 + \gamma_1 A_0\,[\varepsilon_{xx}^2 + {}^1/_4\varepsilon_{xy}^2 + {}^1/_4\varepsilon_{xz}^2] \qquad (1.2)\\
\sigma_{xy} &= \mu\,(1+\gamma_1 A_0^2 + \gamma_2 A_1)\,\varepsilon_{xy} + \gamma_4 A_0\,[(\varepsilon_{xx}+\varepsilon_{yy})\,\varepsilon_{xy} + {}^1/_2\varepsilon_{xz}\varepsilon_{yz}]
\end{aligned}
$$

The expressions for σ_{yy}, σ_{zz}, σ_{yz} and σ_{zx} are obtained by cyclic interchange of scripts.

However, the equations of motion of the elastic medium are very complicated with relations (1.1) or (1.2). We introduce one simplifying assumption. We consider those elastic media for which the effect of a compression on the shear stresses σ_{xy}, σ_{yz} and σ_{xz} is smaller than the effect on the compressive stresses; i.e. we set $\beta_1 = \beta_3 = 0$ or $\gamma_1 = \gamma_2 = \gamma_4 = 0$ approximately, considering the relation between the shear stresses and the components of the strain tensor to be a linear one.

Let us now turn to the formulation and solution of the problem of diffraction of a plane elastic wave around an arbitrary contour C (Fig. 1), taking the instant of time $t = 0$ for the start of diffraction.

We shall express the components u and v of the displacement vector in terms of the potentials Φ and Ψ of the longitudinal and transverse waves by means of the formulas

$$u = \frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}, \qquad v = \frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x} \qquad (1.3)$$

It can be shown that in the absence of external forces the potentials satisfy the equations

$$a^2\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)-\frac{\partial^2\Phi}{\partial t^2}+F(\Delta\Phi)=0, \qquad a^2=\frac{\lambda+2\mu}{\rho}$$
$$b^2\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\right)-\frac{\partial^2\Psi}{\partial t^2}=0, \qquad b^2=\frac{\mu}{\rho} \tag{1.4}$$

Here ρ is the density of the medium, a and b are the velocities of propagation of the longitudinal and transverse waves, respectively, and

$$F(\Delta\Phi)=(\beta_2/\rho)(\Delta\Phi)^2 \quad \text{or} \quad F(\Delta\Phi)=(\gamma_3/\rho)(\Delta\Phi)^3$$

for relations (1.1) and (1.2), respectively.

Let us consider the problem of the diffraction of an elastic wave or of a weak shock wave at the contour C (Fig. 1), the contour being rigidly connected to the elastic medium. In what follows, we shall restrict ourselves to the present problem for definiteness.

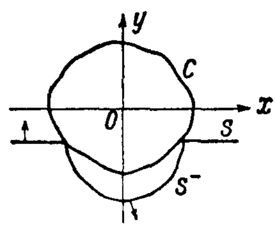

Fig. 1.

For the problem at hand, the boundary conditions for Φ and Ψ take the form

$$\left.\begin{aligned} q_1(\xi_2,t)&=\frac{1}{h_1}\frac{\partial\Phi}{\partial\xi_1}+\frac{1}{h_2}\frac{\partial\Psi}{\partial\xi_2}=\varepsilon(\xi_2,t)\\ q_2(\xi_2,t)&=\frac{1}{h_2}\frac{\partial\Phi}{\partial\xi_2}-\frac{1}{h_1}\frac{\partial\Psi}{\partial\xi_1}=0\end{aligned}\right\} \quad \text{on } C$$

where ξ_1 and ξ_2 are arbitrary orthogonal coordinates related to the contour C, $\xi_1=\xi_{10}$ being the contour itself; h_1 and h_2 are the Lamé coefficients; q_1 and q_2 are the components of the displacement vector of the medium; $\varepsilon(\xi_2, t)$ is the magnitude of the strain of the contour C under the action of the elastic wave. It will be assumed at first that the contour C is absolutely rigid, i.e. $\varepsilon(\xi_2, t)=0$.

In addition, the potentials Φ and Ψ must satisfy the conditions

$$\Phi=\Phi_0, \quad \Psi=\Psi_0 \text{ for } t\leqslant 0, \quad \Phi=\Phi_0, \Psi=\Psi_0 \text{ for } t>0 \tag{1.5}$$

at the front of the reflected wave, where Φ_0 and Ψ_0 are the potentials of the incident elastic wave. Without loss of generality, we shall consider the incident wave to be longitudinal, i.e.

$$\begin{aligned}&\Phi_0=\alpha_0(y-ta), \quad \Psi_0=0 \text{ for } (y-ta)\leqslant 0 \quad \alpha_0=(\Delta\sigma l/(\rho a^2))\\ &\Phi_0=\Psi_0=0 \quad \text{for } (y-ta)>0,\end{aligned} \tag{1.6}$$

Here $\Delta\sigma$ is the jump in stress at the front of the incident wave, $2l$

is the maximum diameter of the contour and ρ is the density of the medium.

Let us introduce the dimensionless variables and parameters

$$x_1 = x / l, \qquad y_1 = y / l, \qquad \tau = at / l, \qquad b_1 = b / a$$
$$\Phi_1 = \Phi / l^2, \qquad \Psi_1 = \Psi / l^2, \qquad \alpha = \alpha_0 / l$$

In what follows we shall omit the subscripts on the dimensionless quantities for simplicity. In equations (1.3) to (1.5) we set

$$\Phi(x, y, \tau) = \Phi_0(x, y, \tau) + \varphi(x, y, \tau), \qquad \Psi(x, y, \tau) = \psi(x, y, \tau) \tag{1.7}$$

The problem of diffraction then reduces to the determination of the potentials φ and ψ which satisfy the following equations and conditions:

$$\Delta\varphi - \frac{\partial^2\varphi}{\partial\tau^2} + F(\Delta\varphi) = 0, \qquad \Delta\psi - \frac{1}{b^2}\frac{\partial^2\psi}{\partial\tau^2} = 0$$
$$\frac{\partial\varphi}{\partial\xi_1} = -\alpha\frac{\partial y}{\partial\xi_1} - \frac{h_1}{h_2}\frac{\partial\psi}{\partial\xi_2}, \qquad \frac{\partial\psi}{\partial\xi_1} = \frac{\alpha h_1}{h_2}\frac{\partial y}{\partial\xi_2} + \frac{h_1}{h_2}\frac{\partial\varphi}{\partial\xi_2} \quad \text{on } C \tag{1.8}$$
$$\varphi = \psi = 0 \quad \text{for } \tau \leqslant 0, \qquad \varphi = \psi = 0 \quad \text{on the reflected wave}$$

We shall solve system (1.8) by setting

$$\varphi(x, y, \tau) = \varphi_1(x, y, \tau) + \varphi_2(x, y, \tau) + \ldots, \psi(x, y, \tau) = \psi_1(x, y, \tau) \tag{1.9}$$

where φ_i are small quantities of ith order.

Substituting (1.9) into (1.8), we obtain two systems for the determination of φ_1, ψ_1 and φ_2, respectively

$$\Delta\varphi_1 - \frac{\partial^2\varphi_1}{\partial\tau^2} = 0, \qquad \Delta\psi_1 - \frac{1}{b^2}\frac{\partial^2\psi_1}{\partial\tau^2} = 0$$
$$\frac{\partial\varphi_1}{\partial\xi_1} = -\alpha\frac{\partial y}{\partial\xi_1} - \frac{h_1}{h_2}\frac{\partial\psi_1}{\partial\xi_2}, \qquad \frac{\partial\psi_1}{\partial\xi_1} = \frac{\alpha h_1}{h_2}\frac{\partial y}{\partial\xi_2} + \frac{h_1}{h_2}\frac{\partial\varphi_1}{\partial\xi_2} \quad \text{on } C \tag{1.10}$$
$$\varphi_1 = \psi_1 = 0 \quad \text{for } \tau \leqslant 0, \qquad \varphi_1 = \psi_1 = 0 \quad \text{on } S^-$$

$$\Delta\varphi_2 - \frac{\partial^2\varphi_2}{\partial\tau^2} + F(\Delta\varphi_1) = 0 \tag{1.11}$$
$$\frac{\partial\varphi_2}{\partial\xi_1} = 0 \quad \text{on } C, \qquad \varphi_2 = 0 \quad \text{for } \tau \leqslant 0, \qquad \varphi_2 = 0 \quad \text{for } \tau > 0 \quad \text{on } S^-$$

As in the case of the problem of diffraction of a weak shock wave [1] at a contour C, the following theorems hold.

Theorem 1.1. Problem (1.10) is equivalent to a mixed Cauchy problem in the space (x, y, τ) or to an auxiliary external problem of flow around an oblique cylinder which is semi-infinite with respect to the axis τ ($\tau \geqslant 0$) and which corresponds to the contour C, the flow consisting of two steady, supersonic streams of ideal gas, $M = \sqrt{2}$, $M = \sqrt{(1 + b^{-2})}$, at the small angle of attack α.

Theorem 1.2. Problem (1.11) is equivalent to a Cauchy problem in the space (x, y, τ) or to the problem of diffraction of a weak shock wave around the contour C in the second approximation [6].

By virtue of Theorem 1.1, using the method of [1], we obtain the following system of integral equations for the values of φ_1 and ψ_1 on the contour C:

$$\varphi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint\limits_{\Sigma}\left[\frac{\varphi_1(x, y, \tau)}{h_1}\frac{\partial V}{\partial\xi_1} - \frac{V}{h_1}\frac{\partial\varphi_1}{\partial\xi_1}\right]d\sigma\right\}$$
$$\psi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint\limits_{\Sigma_1}\left[\frac{\psi_1(x, y, \tau)}{h_1}\frac{\partial V_1}{\partial\xi_1} - \frac{V_1}{h_1}\frac{\partial\psi_1}{\partial\xi_1}\right]d\sigma\right\} \qquad (1.12)$$

where Σ and Σ_1 are the parts of the surface of the cylinder in the auxiliary problem which are cut off by the cones of influence from the point (x_0, y_0, τ_0); the Volterra function is

$$V = \ln\frac{(\tau_0 - \tau) + \sqrt{(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2}}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2}}$$
$$V_1(\tau_0 - \tau, \ldots) = V(b(\tau_0 - \tau), \ldots) \qquad (1.13)$$

Analogously

$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint\limits_{\Sigma}\frac{\varphi_2(x, y, \tau)}{h_1}\frac{\partial V}{\partial\xi_1}d\tau + \iiint\limits_{T}F(\xi, \eta, \zeta)\,Vd\xi d\eta d\zeta\right\} \qquad (1.14)$$

where T is the volume bounded by the surface of the cylinder, the surface of the cone of influence from the point (x_0, y_0, τ_0) and the wave surface of the auxiliary problem.

Problem (1.8) can be generalized [1] to the case of a deformable contour and to the case when

$$\Phi_0(x, y, \tau) = f(y - \tau) \qquad (1.15)$$

The problem of diffraction of an incident transverse wave is solved analogously and, therefore, also that of an arbitrary elastic wave with

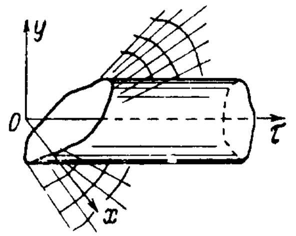

Fig. 2.

$$\Phi_0(x, y, \tau) = \alpha(y - \tau)$$
$$\Psi_0(x, y, \tau) = (\Delta\sigma_1 / \rho a^2)(y - b\tau) \qquad (1.16)$$

1. Let us consider the special case of the problem in which the contour C is a circle of unit radius. In this case the solution of the problem is given by formulas (1.12) and (1.14), where we must set

$$\frac{\partial}{\partial\xi_1} = \frac{\partial}{\partial r}, \qquad \frac{\partial y}{\partial\xi_1} = \sin\theta, \qquad \frac{\partial y}{\partial\xi_2} = \cos\theta \qquad \text{for} \quad r = 1$$

Here (r, θ) are polar coordinates.

It is possible to construct an asymptotic solution of the problem (1.10) to (1.11) for large values of τ for the case of the circle. As a matter of fact, the radius of the reflected wave depends only slightly on θ for large τ, and it is possible to consider approximately that

$$\varphi_1(x, y, \tau) = -\alpha\sin\theta\, f_1(r, \tau), \quad \psi_1(x, y, \tau) = \alpha\cos\theta\, f_2(r, \tau) \qquad (1.17)$$

It can be shown that

$$f_1(r, \tau) = \frac{1}{2\pi i}\int_L C_1(q) K_1(rq) e^{q\tau} dq, \quad f_2(r, \tau) = \frac{1}{2\pi i}\int_L C_2(q) K_1(rqb^{-1}) e^{q\tau} dq \qquad (1.18)$$

$$C_1(q) = \frac{1}{q}\,\frac{-K_1(qb^{-1}) + qb^{-1}K_1'(qb^{-1})}{-K_1(q)K_1(qb^{-1}) + q^2b^{-1}K_1'(q)K_1'(qb^{-1})}$$

$$C_2(q) = \frac{1}{q}\,\frac{-K_1(q) + qK_1'(q)}{-K_1(q)K_1(qb^{-1}) + q^2b^{-1}K_1'(q)K_1'(qb^{-1})}$$

where L is the contour of integration used for the inverse Laplace transform, K_1 is the Bessel function of imaginary argument. Analogously, by setting

$$\varphi_2(x, y, \tau) = \alpha^2[f_3(r, \tau) + \cos 2\theta\, f_4(r, \tau)] \qquad (1.19)$$

we obtain

$$f_3(r, \tau) = \frac{1}{2\pi i}\int_L [A_0(r, q) K_0(rq) + B_0(r, q) I_0(rq)] e^{q\tau} dq \qquad (1.20)$$

$$f_4(r, \tau) = \frac{1}{2\pi i}\int_L [A_1(r, q) K_2(rq) + B_1(r, q) I_2(rq)] e^{q\tau} dq$$

where

$$A_i(r, q) = -\left\{\int_r^\infty (-1)^i F_2(r, q)\, I_{i+1}(rq)\, r dr + \right.$$

$$\left. + \frac{I'_{i+1}(q)}{K'_{i+1}(q)} \int_1^\infty (-1)^i F_2(r, q)\, I_{i+1}(rq)\, r dr - \int_1^\infty (-1)^{i+1} F_2(r, q)\, K_{i+1}(rq)\, r dr \right\}$$

$$B_i(r, q) = -\int_r^\infty (-1)^{i+1} F_2(r, q)\, K_{i+1}(rq)\, r dr \qquad (t = 0, 1)$$

$$F_2(r, q) = \int_0^\infty F_1(r, \tau)\, e^{-q\tau} d\tau, \qquad F(x, y, \tau) = (1 - \cos 2\theta)\, F_1(r, \tau)$$

The remaining parameters of the problem may be determined from the values of φ_1, ψ_1 and φ_2 which have been found. In particular, the magnitudes of the radial stress σ_{rr}, the hoop stress $\sigma_{\theta\theta}$ and the shear stress $\sigma_{\theta r}$ on the circle can be determined. In the case of a deformable circle, it is also possible [6] to construct a solution for large τ.

If the relation between the strain of the circle $\varepsilon(\theta, \tau)$ and $(\sigma_{xx} + \sigma_{yy})$ has the form

$$\varepsilon(\theta, \tau) = k_1 (\sigma_{xx} + \sigma_{yy}) \,/\, 2[(\lambda + \mu)\, \rho a^2] \qquad (1.21)$$

it is necessary to set

$$C_1(q) = \frac{1}{q} \frac{-K_1(qb^{-1}) + qb^{-1} K_1'(qb^{-1})}{-K_1(q) K_1(qb^{-1}) + q^2 b^{-1} K_1'(q) K_1'(qb^{-1}) - k_1 q^3 b^{-1} K_1(q) K_1'(qb^{-1})}$$

$$C_2(q) = \frac{1}{q} \frac{-K_1(q) + qK_1'(q) - k_1 q^2 K_1(q)}{-K_1(q) K_1(qb^{-1}) + q^2 b^{-1} K_1'(q) K_1'(qb^{-1}) - k_1 q^3 b^{-1} K_1(q) K_1'(qb^{-1})}$$

in formulas (1.18). In formulas (1.20), A_i and B_i ($i = 0;\ 1$) are changed in a corresponding manner.

2. Let us now consider the more complicated problem when the contour, in the form of a circle of unit radius, for instance, is in a medium bounded by one or two parallel walls and the front of the incident elastic wave is normal to the walls (Fig. 3).

The effect of the walls can be taken into account by considering, instead of a single circle, a row of circles whose centers are located on a straight line at a distance $2l_1$ apart, the front of the elastic wave being parallel to the line of centers of the circles. We select the distance l_1 in such a way that the effect of the walls is present after the elastic wave passes the circle.

Theorem 1.1 remains in force in this case also, the only difference being that we shall consider an infinite number of cylinders in the auxiliary problem instead of one cylinder, and that the auxiliary problem will be described by system (1.8) or (1.10) to (1.11) with boundary conditions specified on all the cylinders.

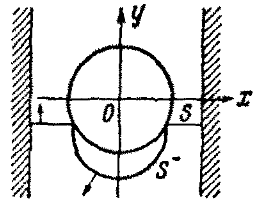

Fig. 3.

The solution of systems (1.10) to (1.11) will have the form

$$\varphi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iint\limits_{\Sigma_i}\left[\varphi_1(x, y, \tau)\frac{\partial V}{\partial r} - \frac{\partial\varphi_1}{\partial r}V\right\} d\sigma\right\}$$

$$\psi_1(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iint\limits_{\Sigma_{i1}}\left[\psi_1(x, y, \tau)\frac{\partial V_1}{\partial r} - \frac{\partial\psi_1}{\partial r}V_1\right] d\sigma\right\} \qquad (1.22)$$

and, analogously

$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iint\limits_{\Sigma_i}\varphi_2(x, y, \tau)\frac{\partial V}{\partial r}\partial\sigma + \right.$$

$$\left. + \iiint\limits_{T_i} F(x, y, \tau)\, V dx\, dy\, d\tau\right\} \qquad (i = 0;\ \pm 1;\ \pm 2; \ldots) \qquad (1.23)$$

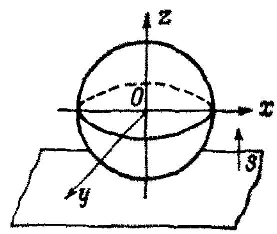

Fig. 4.

Here Σ_i and Σ_{i1} are the parts of the surfaces of the cylinders which are cut off by the cones of influence from the point (x_0, y_0, τ_0), and T_1 is the corresponding volume, which is bounded by the surface of the cone of influence, the surfaces of the cylinders Σ_i and the common wave surface.

It is easy to see that in (1.22) and (1.23) the φ_1, ψ_1 and φ_2 under the integral are unknown only on the surfaces Σ_0 and Σ_{01}, since the values of the functions φ_1, ψ_1 and φ_2 on the remaining surfaces coincide with values on the surfaces Σ_0 and Σ_{01} found for earlier values of the dimensionless time τ. Solutions (1.22) and (1.23) can also be generalized to the case of a deformable contour.

2. Formulation and solution of the three-dimensional problem of diffraction of a plane wave. This problem can be solved under quite general assumptions. For simplicity of exposition we shall consider the problem of diffraction of an elastic wave around a sphere of unit radius (Fig. 4) under the limitations explained at the start of Section 1.

We shall formulate the problem in spherical coordinates (r, θ, ϑ). By virtue of the symmetry of the problem, it is obvious that the unknown functions do not depend on the angle ϑ. The displacement vector has only the two components

$$q_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \Psi), \qquad q_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial r}(r\Psi) \tag{2.1}$$

The potentials Φ and Ψ satisfy the equations

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) &= \frac{\partial^2 \Phi}{\partial \tau^2} - F(\Delta\Phi) \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Psi}{\partial \theta}\right) &= \frac{\partial^2 \Psi}{\partial \tau^2}\frac{1}{b^2} \end{aligned} \tag{2.2}$$

Assuming that the diffraction of the elastic wave around the sphere begins at $\tau = 0$, and setting

$$\Phi = \Phi_0 + \varphi_1 + \varphi_2 + \ldots, \qquad \Psi = \psi_1, \qquad \Phi_0 = \alpha(z - \tau) \tag{2.3}$$

we reduce the problem to solution of wave equations for φ_1, ψ_1 and φ_2 with the boundary and initial conditions

$$\begin{aligned} &\frac{\partial^2 \varphi_1}{\partial \tau^2} = \Delta\varphi_1, \qquad \frac{\partial^2 \varphi_2}{\partial \tau^2} = \Delta\varphi_2 + F(\Delta\varphi_1), \qquad \frac{\partial^2 \varphi_1}{\partial \tau^2} = \Delta\psi_1 b^2 \\ &\frac{\partial \varphi_1}{\partial r} = -\alpha\cos\theta - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\psi_1), \qquad \frac{\partial \varphi_2}{\partial r} = 0 \\ &\frac{\partial \psi_1}{\partial r} = -\alpha\sin\theta + \frac{\partial \varphi_1}{\partial \theta} - \psi_1 \quad \text{for } r = 1 \\ &\varphi_1 = \varphi_2 = \psi_1 = 0 \quad \text{for } \tau \leqslant 0, \qquad \varphi_1 = \varphi_2 = \psi_1 = 0 \quad \text{for } \tau > 0 \quad \text{on } S^- \end{aligned} \tag{2.4}$$

Thus, we have the following theorem.

Theorem 2.1. The problem of the diffraction of an elastic wave around a sphere, described by system (2.4), is equivalent [6] to a mixed Cauchy problem or to an auxiliary problem of flow around the corresponding four-dimensional oblique cylinder which is semi-infinite with respect to the τ-axis, the flows consisting of two steady supersonic streams of ideal gas, $M = \surd 2$ and $M = \surd(1 + b^{-2})$, at the small angle of attack α.

Solving the auxiliary problem by a method analogous to the method of Volterra for the three-dimensional case, we obtain

$$\begin{aligned} \varphi_1(1, \theta_0, \tau_0) &= \frac{1}{4\pi}\frac{\partial^2}{\partial \tau_0^2}\left\{\iiint\limits_{T_0}\left[\varphi_1(1, \theta, \tau)\frac{\partial V^1}{\partial r} - \frac{\partial \varphi_1}{\partial r}V^1\right] d\theta\, d\tau\, d\vartheta\right\} \\ \varphi_2(1, \theta_0, \tau_0) &= \frac{1}{4\pi}\frac{\partial^2}{\partial \tau_0^2}\left\{\iiint\limits_{T_0}\varphi_2(1, \theta, \tau)\frac{\partial V^1}{\partial r}\, d\theta\, d\tau\, d\vartheta + \right. \\ &\quad \left. + \iiiint\limits_{Q} F(r, \theta, \tau)\, V^1\, dr\, d\theta\, d\tau\, d\vartheta\right\} \\ \psi_1(1, \theta_0, \tau_0) &= \frac{1}{4\pi}\frac{\partial^2}{\partial \tau_0^2}\left\{\iiint\limits_{T_1}\left[\psi_1(1, \theta, \tau)\frac{\partial V_1^1}{\partial r} - \frac{\partial \psi_1}{\partial \tau}V_1^1\right] d\theta\, d\tau\, d\vartheta\right\} \end{aligned} \tag{2.5}$$

Here T_0 and T_1 are the parts of the surface of the four-dimensional cylinder cut off by the cones of influence [4] from the point $(1, \theta_0, \tau_0, \vartheta_0)$ [6]; Q is the four-dimensional volume bounded by the surface of the cone of influence, the surface T_0 and the wave surface; the function

$$V^1 = 1 - (\tau_0 - \tau) / \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$$

$$V_1^1 (\tau_0 - \tau, \ldots) = V^1 (b (\tau_0 - \tau), \ldots) \tag{2.6}$$

As in the two-dimensional problem, an asymptotic solution of problem (2.4) can be found for large τ. It can be shown that for large values of τ

$$\varphi_1 (r, \theta, \tau) = -\frac{\alpha \cos \theta}{2\pi i \sqrt{r}} \int_L C_3 (q) K_{3/2} (rq) e^{q\tau} dq$$

$$\psi_1 (r, \theta, \tau) = -\frac{\alpha \sin \theta}{2\pi i \sqrt{r}} \int_L C_4 (q) K_{3/2} (rqb^{-1}) e^{q\tau} dq \tag{2.7}$$

where

$$C_3 (q) = \frac{1}{q} \frac{qb^{-1} K_{3/2}' (q) - K_{3/2} (qb^{-1})}{q^2 b^{-1} K_{3/2}' (q) K_{3/2}' (qb^{-1}) + q K_{3/2}' (q) K_{3/2} (qb^{-1}) - 2 K_{3/2} (q) K_{3/2} (qb^{-1})}$$

$$C_4 (q) = \frac{1}{q} \frac{q K_{3/2}' (q) - K_{3/2} (q)}{q^2 b^{-1} K_{3/2}' (q) K_{3/2}' (qb^{-1}) + q K_{3/2}' (q) K_{3/2} (qb^{-1}) - 2 K_{3/2} (q) K_{3/2} (qb^{-1})}$$

The solution for the function $\varphi_2(r, \theta, \tau)$ can be written out analogously. Problem (2.4) can be generalized [1] to the case of a deformable sphere and for

$$\Phi_0 (x, y, z, \tau) = f (z - \tau) \tag{2.8}$$

3. On the theory of diffraction of plane electromagnetic waves. Nonsteady problems of diffraction of plane electromagnetic waves around convex, perfectly conducting bodies of revolution will be considered, when the wave is propagating along the axis of revolution. We shall assume that a plane electromagnetic wave is incident on a perfectly conducting body which is surrounded by a homogeneous dielectric with the parameters ε^0 and μ^0. The wave excites surface currents on the surface of the body; these, in turn, become the source of the scattered or diffracted field.

The electric and magnetic intensities $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations

$$\operatorname{curl} \mathbf{H} = \frac{\varepsilon^0}{c} \frac{\partial \boldsymbol{E}}{\partial t}, \qquad \operatorname{div} \mathbf{H} = 0, \qquad \operatorname{curl} \mathbf{E} = -\frac{\mu^0}{c} \frac{\partial \boldsymbol{H}}{\partial t}, \qquad \operatorname{div} \mathbf{E} = 0 \tag{3.1}$$

and the boundary condition

$$\mathbf{n} \times [\mathbf{E}_1 + \mathbf{E}_2] = 0 \tag{3.2}$$

on the surface of the perfect conductor, where $\mathbf{n}$ is the normal to the body; $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric intensities of the incident and scattered waves, respectively, on the surface of the body; and c is the speed of light.

In view of the symmetry of the problem relative to the axis of rotation, we obtain

$$E_{\xi_1} = \frac{1}{h_2 h_3} \frac{\partial}{\partial \xi_2} \left[\frac{h_3}{h_1 h_2} \frac{\partial}{\partial \xi_2} (h_2 \Pi) \right], \qquad E_{\xi_2} = -\frac{1}{h_1 h_3} \frac{\partial}{\partial \xi_1} \left[\frac{h_3}{h_1 h_2} \frac{\partial}{\partial \xi_2} (h_2 \Pi) \right]$$
$$E_{\xi_3} = H_{\xi_1} = H_{\xi_2} = 0, \qquad H_{\xi_3} = \frac{\varepsilon^\circ}{c} \frac{1}{h_1 h_2} \frac{\partial^2 (h_2 \Pi)}{\partial \xi_2 \partial t} \tag{3.3}$$

where the function Π is called the Hertz potential.

In the derivation of relations (3.3) the Hertz vector G is taken as

$$G = \{0, e_2 \Pi, 0\} \tag{3.4}$$

If

$$G = \{e_1 \Pi, 0, 0\} \tag{3.5}$$

then $h_1 \Pi$ should appear in the parentheses in (3.3) instead of $h_2 \Pi$.

Substituting (3.3) into (3.1), we obtain the wave equation in general orthogonal curvilinear coordinates ξ_1, ξ_2 and ξ_3 for the potential Π

$$\frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \xi_1} \left(\frac{h_1 h_2 h_3}{h_1^2} \frac{\partial \Pi}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{h_1 h_2 h_3}{h_2^2} \frac{\partial \Pi}{\partial \xi_2} \right) \right] = \frac{\varepsilon^\circ \mu^\circ}{c^2} \frac{\partial^2 \Pi}{\partial t^2} \tag{3.6}$$

Here h_1, h_2 and h_3 are the Lamé coefficients.

Since the body is one of revolution

$$h_1 = h_1(\xi_1, \xi_2), \qquad h_2(\xi_1, \xi_2) = h_2, \qquad h_3 = h_3^1(\xi_1, \xi_2) / \sqrt{1 - \xi_3^2}$$

the coefficients of equation (3.6) do not depend on ξ_3, and, therefore, the function $\Pi = \Pi(\xi_1, \xi_2)$. Condition (3.2) assumes the form

$$\frac{\partial}{\partial \xi_1} \left[\frac{h_3}{h_1 h_2} \frac{\partial}{\partial \xi_2} (h_2 \Pi) \right] = 0 \quad \text{for } \xi_1 = \xi_{10} \tag{3.7}$$

where $\xi_1 = \xi_{10}$ is the equation of the surface of the body.

The potential Π also satisfies the conditions

$$\Pi = \Pi_0(x, y, z, t) \quad \text{for } t \leqslant 0, \qquad \Pi = \Pi_0(x, y, z, t) \quad \text{for } t > 0 \tag{3.8}$$

on the boundary of the scattered or diffracted field. We set

$$\Pi(x, y, z, t) = \Pi_0(x, y, z, t) + \varphi(x, y, z, t) \tag{3.9}$$

where Π_0 is the potential of the incident field, and substitute (3.9) into equation (3.6) and into conditions (3.7) and (3.8). We then obtain the following system to determine the perturbed potential $\varphi(x, y, z, t)$:

$$\begin{gathered}
\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3h_1}{h_1^2}\frac{\partial\varphi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_1h_2h_3}{h_2^2}\frac{\partial\varphi}{\partial\xi_2}\right)\right] = \frac{\varepsilon^\circ\mu^\circ}{c^2}\frac{\partial^2\varphi}{\partial t^2} \\
\frac{\partial\varphi}{\partial\xi_1} = -\frac{\partial\Pi_0}{\partial\xi_1} - \frac{1}{h_2}\int\left\{h_2\frac{\partial}{\partial\xi_1}\left(\ln\frac{h_3}{h_1}\right)\frac{\partial\Pi}{\partial\xi_2} + \frac{h_1h_2}{h_3}\frac{\partial}{\partial\xi_1}\left(\frac{h_3}{h_1}\ln h_2\right)\Pi\right\}d\xi_2 \\
\text{for } \xi_1 = \xi_{10} \\
\varphi = 0 \quad \text{for } t \leqslant 0, \qquad \varphi = 0 \quad \text{on } S^-
\end{gathered} \tag{3.10}$$

Let us consider some special cases.

1. **Diffraction of a plane electromagnetic wave around an infinite circular cylinder.** We shall assume that the front of the incident electromagnetic wave is parallel to the z-axis. Setting, for simplicity

$$\Pi_0 = \alpha\left(y - \frac{c}{\sqrt{\varepsilon^\circ\mu^\circ}}\,t\right) \tag{3.11}$$

we reduce system (3.10) to the form

$$\begin{gathered}
\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{\varepsilon^\circ\mu^\circ}{c^2}\frac{\partial^2\varphi}{\partial t^2} \\
\frac{\partial\varphi}{\partial r} = -\frac{\alpha}{r_0}\sin\theta \ \text{ for } r = r_0 \\
\varphi = 0 \quad \text{for } t \leqslant 0, \qquad \varphi = 0, \quad \text{for } t > 0 \text{ on } S^-
\end{gathered} \tag{3.12}$$

Here α is some small quantity introduced into the solution of the problem for convenience and which can be eliminated at the end; r and θ are polar coordinates.

It is easy to see that system (3.12) also describes the problem of diffraction of a weak shock wave around a circle [1]. Therefore, [1] $\varphi(r, \theta, t)$ on the circle satisfies the linear integral equation

$$\varphi(1, \theta_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iiint_{\Sigma}\left[\varphi(1, \theta\ \tau)\frac{\partial V}{\partial r_1} - \frac{\partial\varphi}{\partial r_1}V\right]\partial\sigma\right\} \tag{3.13}$$

$$r_1 = r / r_0, \qquad \tau = at / r_0, \qquad a = c / \sqrt{\mu^\circ\varepsilon^\circ}$$

where Σ is the surface of the cylinder in the auxiliary problem cut off by the cone of influence from the point $(r/r_0 = 1,\ \theta_0,\ \tau_0)$.

For large values of τ (or of t)

$$\varphi(r_1, \theta, \tau) = \frac{1}{2\pi i}\int_L \frac{K_1(r_1 q)}{K_1'(q)} e^{q\tau} \frac{dq}{q^2} \tag{3.14}$$

If

$$\Pi_0(x, y, z, t) = f(y/r_0 - \tau) \tag{3.15}$$

then, using the Duhamel integral, we obtain

$$\varphi_1(r_1, \theta, \tau) = f(0)\varphi(r_1, \theta, \tau) + \int_0^\tau f'(\xi)\varphi(r_1, \theta, \tau - \xi)\, d\xi \tag{3.16}$$

where φ_1 is the Hertz potential in case (3.15). For large values of τ

$$\varphi_1(r_1, \theta, \tau) = \frac{\sin\theta}{2\pi i}\int_L \left[f(0) + \int_0^\tau f'(\xi)\, e^{q\xi} d\xi \right] \frac{K_1(r_1 q)}{K_1(q)}\ e^{q\tau} \frac{dq}{q^2} \tag{3.17}$$

In the case in which the front of the incident wave forms an angle γ with the z-axis, the problem reduces to that solved above [6] where the quantity τ must be replaced by the quantity

$$\eta = \tan^{-1}\gamma\,(\tau \sin^{-1}\gamma - z)$$

2. *Diffraction of a plane electromagnetic wave around a sphere.* We shall consider that the incident electromagnetic wave is propagating along the z-axis in the positive direction (Fig. 4). In dimensionless spherical coordinates $(r,\ \theta,\ \vartheta)$ the perturbed Hertz potential φ satisfies the system

$$\frac{1}{r^2}\ \frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\ \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) = \frac{\partial^2\varphi}{\partial\tau^2} \tag{3.18}$$

$$\frac{\partial\varphi}{\partial r} = -\alpha\cos\theta - (\Pi_0 + \varphi) \quad \text{for } r = 1$$

$$\varphi = 0 \quad \text{for } \tau \leqslant 0 \quad \text{on } S^-$$

Here it is also assumed that

$$\Pi_0(x, y, z, \tau) = \alpha\ (z - \tau)$$

It is not difficult to see that problem (3.18) is equivalent to the problem of diffraction of a weak shock wave around the sphere [6] and, therefore

$$\varphi(1, \theta_0, \tau_0) = \frac{1}{4\pi} \frac{\partial^2}{\partial \tau_0^2} \left\{ \iiint\limits_T \left[\varphi(1, \theta, \tau) \frac{\partial V^1}{\partial r} - \frac{\partial \varphi}{\partial r} V^1 \right] d\theta d\tau d\vartheta \right\} \tag{3.19}$$

where V^1 has the form (2.6); T is the hypersurface of the appropriate four-dimensional cylinder $r = 1$ which is cut off by the cone of influence from the point $(r = 1,\ \theta_0,\ \vartheta_0,\ \tau_0)$.

If case (3.5) is considered, the second condition in system (3.16) takes the form

$$\frac{\partial \varphi}{\partial r} = -\alpha \cos\theta \qquad \text{for } r = 1$$

and problem (3.18) is identical to the problem of diffraction of a weak plane shock wave around the sphere. For large values of τ we may set

$$\varphi - \alpha\tau = -2\alpha \cos\theta f_1^\circ(r, \tau) \tag{3.20}$$

It can be shown that

$$f_1^\circ(r, \tau) = \frac{1}{2\pi i \sqrt{r}} \int\limits_L \frac{K_{3/2}(rq) e^{q\tau}}{q K_{3/2}'(q) + (q^{-1} - 1/2) K_{3/2}(q)} \frac{dq}{q} \tag{3.21}$$

In case (3.5), we set

$$\varphi(r, \theta, \tau) = -\alpha \cos\theta f_2^\circ(r, \tau)$$

$$f_2^\circ(r, \tau) = \frac{1}{2\pi i \sqrt{r}} \int\limits_L \frac{K_{3/2}(rq) e^{q\tau}}{q K_{3/2}'(q) - 1/2 K_{3/2}(q)} \frac{dq}{q} \tag{3.22}$$

Problem (3.18) can be generalized to the case

$$\Pi_0(x, y, z, \tau) = f(z - \tau)$$

3. *Solution of system (3.10). Theorem 3.1.* The general problem of diffraction of a plane electromagnetic wave around an arbitrary convex body of revolution is equivalent to the problem of flow around the corresponding semi-infinite four-dimensional cylinder $\xi_1 = \xi_{10}$, the flow consisting of a supersonic stream of ideal gas, $M = \sqrt{2}$, at a small angle of attack.

We obtain an integral equation of the form

$$\varphi(x_0, y_0, z_0, \tau_0) = \frac{1}{4\pi} \frac{\partial^2}{\partial \tau_0^2} \left\{ \iiint\limits_T \left[\frac{\varphi(x, y, z, \tau)}{h_1} \frac{\partial V^1}{\partial \xi_1} - \frac{V^1}{h_1} \frac{\partial \varphi}{\partial \xi_1} \right] d\xi_2 d\xi_3 d\tau \right\} \tag{3.23}$$

for the potential $\varphi(x, y, z, t)$ on the surface of the arbitrary body of revolution. Explicit asymptotic expressions for φ at large τ can be obtained only for the circular cylinder and the sphere.

Analogously, problems of diffraction of plane electromagnetic waves around bodies of arbitrary shape (or around a group of bodies) can be solved by reducing them to the corresponding mixed Cauchy problems in the space (x, y, z, τ).

Note. Problems of diffraction of elastic and electromagnetic waves reduce to the solution of the corresponding singular integro-differential equations. To solve these equations numerically it is more convenient, for instance in equations (1.12), (1.14), (1.22), etc., to introduce the functions

$$K_0(\tau_0-\tau,\ x_0-x,\ y_0-y)=\int\frac{dV}{\partial\xi_1}\,d\tau, \qquad K_1(\tau_0-\tau,\ x_0-x,\ y_0-y)=\int K\,d\tau$$

The function K is continuous on Σ and goes to zero along the entire characteristic part of the boundary of the region Σ; the function K_1 is continuous on Σ and goes to zero at the point (x_0, y_0, τ_0). Therefore, by covering the region Σ with a sufficiently fine net, it is possible either to eliminate the neighborhood of the point (x_0, y_0, τ_0) or to approximate the integral expressions in the neighborhood of (x_0, y_0, τ_0) in some manner and to reduce the numerical solution of the integro-differential equations to simple quadratures over the remaining part of Σ.

The uniqueness of the solution of the given integro-differential equations follows from their very construction, or can be rigorously proved by the usual methods. The existence of the solution is proved by the method of Hadamrd.

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